APPROXIMATION AND COMPARISON OF ORDINARY DIFFERENTIAL EQUATION USING BY NEW ITERATION METHODS

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Abstract: *In this study, Modified Ishikawa, Modified Krasnoselskii, Extra Modified Ishikawa and New Modified Ishikawa Iteration methods are presented. The approximate solution of the different type of the problems is solved subject to the initial condition. Then comparisons of these methods with exact solutions are considered. Finally some numerical examples are introduced and related tables are given and the graphs are sketched.*

Keywords: *Ordinary Differential Equation, Euler Method, Fixed Point, Numerical Analysis, Modified Ishikawa Iteration, Modified Krasnoselskii Iteration, Picard successive Iteration Method*

2012MSC: *65K15, 65L07, 65L06, 65L70.*

1. INTRODUCTION

The most of the academician studied many kinds of the numerical methods which are used to solve different types of differential equations up to now [19-24]. Namely L.E.J. Brouwer first presented the fixed point theory on normed linear space [8]. Later on this, there has been growing interest in approximation of fixed point theory on normed linear spaces [1,4,8,9], Banach spaces [2,10,12,14,16,18], and Hilbert spaces [11,17], respectively.

In one of our previously studies, we have dealt with Modified Ishikawa iteration to solve the different type of differential equations and compared the results with the method of the Runga-Kutta, Euler, and Picard [6]. On the other hand, we recently worked on Modified Krasnoselskii method to solve some of the differential equations and at the end, compared the results with the method of Runga-Kutta, Euler, and Picard [5].

In this work, firstly we picked different type of differential equations and showed that how to applied New Modified Ishikawa iteration method and Extra Modified Ishikawa iteration method to the given problems. And later on, we considered the comparisons between the Modified Ishikawa, Modified Krasnoselskii, New Modified Ishikawa and Extra Modified Ishikawa iteration methods each other. And following this, we gave the tables and sketched the graphs. Consequently, we decided that which one of these iterations is more powerful or the best approximation.

Now, Let us give some of the important theorems and definitions.

Theorem 1.1. (Banach constraction principle) Let (X, d) be a complete metric space and $T: X \to X$ be a constraction with the Lipschitzian constant L. Then T has a unique fixed point $u \in X$.

Furthermore, for any $x \in X$ we have $\lim_{n\to\infty} T^n(x) = u$ with $d(T^n(x), u) \le \frac{L^n}{1-L} d(x, T(x))$ [1]

Corollary 1.2. Let (X, d) be a complete metric space and let $B(x_0, r) = \{x \in X: d(x, x_0) < r\}$, where $x_0 \in X$ and $r > 0$. Suppose $T: B(x_0, r) \to X$ is a constaction (that is, $d(T(x), T(y)) \le L d(x, y)$ for all $x, y \in B(x_0, r)$ with $0 \le L < 1$) with $d(T(x_0), x_0) < (1 - L)r$. Then T has a unique fixed point in $B(x_0, r)$ [1].

Definition 1.3. Let X be a normed linear space and $T : x \to x$ a given operator. If the sequence $\{x_n\}_{n=0}^{\infty}$ provides the condition $x_{n+1} = T x_n$ for $n = 0,1,2, \dots$, then this is called the Picard iteration [7,18].

Definition 1.4. Let X be a normed linear space and $T : x \to x$ a given operator. If $x_0 \in X$, $\lambda \in [0,1]$ and also the $\{x_n\}_{n=0}^{\infty}$ sequence provides the condition

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n \quad \text{for } n = 0,1,2, \dots$$

then this is called the Krasnoselskii iteration [3].

Definition 1.5. Let $x_0 \in X$ be arbitrary. If the sequence $\{x_n\}_{n=0}^{\infty}$ provides the condition

$$x_{n+1} = (1 - \alpha_n)x_n + \{\alpha_n\}T y_n$$
$$y_n = (1 - \beta_n)x_n + \{\beta_n\}T x_n$$

for $n = 0,1,2, \dots$, then this is called the Ishikawa iteration where (α_n) and $\{\beta_n\} \subset [0,1]$ is sequences of positive numbers that satisfy the following conditions:

(i) $0 \le \alpha_n \le \beta_n < 1$ for all positive integers n

(ii) $\lim_{n\to\infty} \beta_n = 0$

(iii) $\sum_{n\ge 0} \alpha_n \beta_n = \infty$. [12,13]

Definition 1.6. If $y_0 \in X$, $\lambda \in [0,1]$ and T is defined contraction mapping regard as Picard iteration and also the $\{y_n\}_{n=0}^{\infty}$ sequence provides the conditions

$$y_{n+1} = y_0 + \int_{x_0}^{x} F(t, y_n(t))dt \quad \text{for } n = 0,1$$
$$y_{n+1} = (1 - \lambda)y_n + \lambda T y_{n-2} \quad \text{for } n = 2,3, \dots$$
$$T y_{n-1} = y_n \quad , \quad 0 < \lambda < 1$$

, then this is called the Modified Krasnoselskii iteration where $T = \int_{x_0}^{x} F(t, y_n(t))dt$ and $y'(t) = F(t, y(t))$ [5].

Definition 1.7. If $y_0 \in X$, $\lambda \in [0,1]$, $\gamma \in [0,1]$ and T is defined contraction mapping with regard to Picard iteration and also the $\{y_n\}_{n=0}^{\infty}$ sequence provides the conditions

$$\left.\begin{aligned} y_{n+1} &= \lambda y_{n-1} + (1 - \lambda)T y_{n-1} \\ y_n &= (1 - \gamma)y_{n-2} + \gamma T y_{n-2} \end{aligned}\right\} \text{for } n = 2,4, \dots$$
$$y_{n+1} = y_0 + \int_{x_0}^{x} F(t, y_n(t))dt \quad \text{for } n = 0$$
$$T y_{n-1} = y_n \quad , \quad 0 < \lambda, \gamma < 1$$

, then this is called the Modified Ishikawa iteration where $T = \int_{x_0}^{x} F(t, y_n(t))dt$ and $y'(t) = F(t, y(t))$[6].

Definition 1.8. If $y_0 \in X$, $\lambda \in [0,1]$, $\gamma \in [0,1]$ and T is defined contraction mapping with regard to Picard iteration and also the $\{y_n\}_{n=0}^{\infty}$ sequence provides the conditions

$$\left.\begin{aligned} y_{n+1} &= (1 - \lambda)y_{n-1} + \lambda T y_{n-1} \\ y_n &= \gamma y_{n-2} + (1 - \gamma)T y_{n-2} \end{aligned}\right\} \text{for } n = 2,4, \dots$$
$$y_{n+1} = y_0 + \int_{x_0}^{x} F(t, y_n(t))dt \quad \text{for } n = 0$$
$$T y_{n-1} = y_n \quad , \quad 0 < \lambda, \gamma < 1$$

then this is called the New Modified Ishikawa Iteration where $T = \int_{x_0}^{x} F(t, y_n(t))dt$ and $y'(t) = F(t, y(t))$

Definition 1.9. If $y_0 \in X$, $\lambda \in [0,1]$, $\gamma \in [0,1]$ and T is defined contraction mapping with regard to Picard iteration and also the $\{y_n\}_{n=0}^{\infty}$ sequence provides the conditions

$$\left.\begin{aligned} y_{n+1} &= \lambda y_{n-1} + (1-\lambda)Ty_{n-1} \\ y_n &= \gamma y_{n-2} + (1-\gamma)Ty_{n-2} \end{aligned}\right\} \text{ for } n = 2,4,\ldots$$

$$y_{n+1} = y_0 + \int_{x_0}^{x} F(t, y_n(t))dt \quad \text{for } n = 0$$

$$Ty_{n-1} = y_n \quad , \quad 0 < \lambda, \gamma < 1$$

, then this is called the Extra Modified Ishikawa Iteration where $T = \int_{x_0}^{x} F(t, y_n(t))dt$ and $y'(t) = F(t, y(t))$

2. APPLICATION OF METHODS

Example 2.1. Let us consider the differential equation $y' = \sqrt{|y|}$ subject to the initial condition $y(0) = 1$ [15]. Firstly, we obtained the exact solution of the equation as $|y| = \frac{1}{4}(x+2)^2 = 1 + x + \frac{x^2}{4}$. By Theorem 1.1 and Corollary 1.2, since $T = \int_{x_0}^{x} F(t, y_n(t))dt$ and $y'(t) = F(t, y(t))$, then

$$|T(x) - T(y)| = \left|\int_0^x \sqrt{t}dt - \int_0^y \sqrt{t}dt\right| = \left|\frac{2}{3}\sqrt{t^3}\Big|_0^x - \frac{2}{3}\sqrt{t^3}\Big|_0^y\right| \le \frac{2}{3}\left|\sqrt{x^3} - \sqrt{y^3}\right| \le \frac{2}{3}|x-y|.$$

So, $|T(x) - T(y)| \le \frac{2}{3}|x-y|$ is found. Thus T has a unique fixed point, which is the unique solution of the integral equation $T = \int_{x_0}^{x} F(t, y_n(t))dt$ or the differential equation $y' = \sqrt{|y|}$,

Firstly, we consider the approximate solution using by the Picard iteration method. Thus

$$y_1 = 1 + x$$

$$y_2 = \frac{1}{3} + \frac{2}{3}(1+x)^{3/2}$$

are obtained. If we take the series expansion of the function $(1+x)^{3/2}$ for the seven terms, then

$$y_2 = 1 + x + \frac{x^2}{4} + \frac{x^3}{24} + \frac{x^4}{64} + \frac{x^5}{128} + \frac{7x^6}{1536}$$

is found. Now we calculate the approximate solution by the Euler method. At first we use the formula

$$y_{n+1} = y_n + hF(x_n, y_n)$$

with $F(x,y) = \sqrt{|y|}$, $h = 0.2$ and $x_0 = 0$, $y_0 = 1$ from the initial condition $y(0) = 1$,

we have $F(0,1) = 1$. We now proceed with the calculations as follows:

$$F_0 = F(0,1) = 1$$

$$y_1 = y_0 + hF_0 = 1 + 0.2 = 1.2$$

$$F_1 = F(0.2, 1.2) = 1.095445115$$

$$y_2 = 1.2 + 0.2F_1 = 1.419089023$$

$$F_2 = F(0.4, 1.419089023) = 1.19125523$$

$$y_3 = 1.419089023 + 0.2F_2 = 1.657340069$$

Finally, applying the Runga-Kutta method to the given initial value problem, we carry out the intermediate calculations in each step to give figures after the decimal point and round off the final results at each step to four such places.

Here $F(x,y) = \sqrt{|y|}$, $x_0 = 0$, $y_0 = 1$ and $h = 0.2$ .Using these quantities, we calculated successively k_1, k_2, k_3, k_4 and K_0 defined by

$$k_1 = hF(x_0, y_0)$$

$$k_2 = hF\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hF\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hF(x_0 + h, y_0 + k_3)$$

and $K_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, $y_{n+1} = y_n + K_0$ .Thus we find k_1, k_2, k_3, k_4 for $n = 0$ as follows:

$k_1 = 0.2$
$k_2 = 0.209761769$
$k_3 = 0.210226628$
$k_4 = 0.220020601$

So, $y_1 = 1.209999565$ is obtained for $x_1 = 0.2$. On the other hand, we calculated k_1, k_2, k_3, k_4 for $n = 1$ as follows:

$k_1 = 0.21999996$
$k_2 = 0.229782466$
$k_3 = 0.230207801$
$k_4 = 0.240017279$

Hence $y_2 = 1.4399999194$ is calculated for $x_2 = 0.4$. Finally we get k_1, k_2, k_3, k_4 for $n = 2$ as follows:

$k_1 = 0.239999993$
$k_2 = 0.249799913$
$k_3 = 0.2250131316$
$k_4 = 0.260014756$

Thus, $y_3 = 1.689999654$ is obtained for $x_3 = 0.6$. Now, applying the New Modified Ishikawa Iteration Method to the equation for different value of λ and γ, then

$$\left.\begin{aligned} y_1 &= 1 + x \\ y_2 &= 1 + 0.5x \\ y_3 &= 1 + 0.75x \\ y_4 &= 1 + 0.625x \\ y_5 &= 1 + 0.6875x \\ y_6 &= 1 + 0.65625x \\ y_7 &= 1 + 0.671875 \end{aligned}\quad \text{for} \right\} \lambda = 0.5 \text{ and } \gamma = 0.5$$

$$\left.\begin{aligned} y_1 &= 1 + x \\ y_2 &= 1 + 0.75x \\ y_3 &= 1 + 0.875x \\ y_4 &= 1 + 0.84375x \\ y_5 &= 1 + 0.859375x \\ y_6 &= 1 + 0.85546875x \\ y_7 &= 1 + 0.857421875x \end{aligned}\quad \text{for} \right\} \lambda = 0.5 \text{ and } \gamma = 0.25$$

$$\left.\begin{aligned} y_1 &= 1 + x \\ y_2 &= 1 + 0.5x \\ y_3 &= 1 + 0.875x \\ y_4 &= 1 + 0.6875x \\ y_5 &= 1 + 0.828125x \\ y_6 &= 1 + 0.7578125x \\ y_7 &= 1 + 0.810546875x \end{aligned}\quad \text{for} \right\} \lambda = 0.25 \text{ and } \gamma = 0.5$$

are obtained, and also Extra Modified Ishikawa Iteration Method to the equation for different value of λ and γ, then

$$\left.\begin{array}{l} y_1 = 1 + x \\ y_2 = 1 + 0.5x \\ y_3 = 1 + 0.75x \\ y_4 = 1 + 0.625x \\ y_5 = 1 + 0.6875x \\ y_6 = 1 + 0.65625x \\ y_7 = 1 + 0.671875x \end{array} \quad \text{for} \right\} \lambda = 0.5 \text{ and } \gamma = 0.5$$

$$\left.\begin{array}{l} y_1 = 1 + x \\ y_2 = 1 + 0.75x \\ y_3 = 1 + 0.875x \\ y_4 = 1 + 0.84375x \\ y_5 = 1 + 0.859375x \\ y_6 = 1 + 0.85546875x \\ y_7 = 1 + 0.857421875x \end{array} \quad \text{for} \right\} \lambda = 0.5 \text{ and } \gamma = 0.25$$

$$\left.\begin{array}{l} y_1 = 1 + x \\ y_2 = 1 + 0.5x \\ y_3 = 1 + 0.625x \\ y_4 = 1 + 0.5625x \\ y_5 = 1 + 0.578125x \\ y_6 = 1 + 0.5703125x \\ y_7 = 1 + 0.572265625x \end{array} \quad \text{for} \right\} \lambda = 0.25 \text{ and } \gamma = 0.5$$

are calculated. Now, as it is seen [5] and [6] we compared the results that we already obtained with the methods of Picard, Euler and Runga-Kutta. Let us give the table of the absolute error with respect to the Example 2.1 for different value of λ and γ as follows:

Table 1 Absolute Error

		$x = 0.2$	$x = 0.4$	$x = 0.6$
Modified Ishikawa Iteration	$\lambda = 0.5, \gamma = 0.5$	0.12398437	0.173203126	0.289804688
	$\lambda = 0.5, \gamma = 0.25$	0.129110108	0.278220216	0.447330323
	$\lambda = 0.25, \gamma = 0.5$	0.09571167	0.21142334	0.34713501
	$\lambda = 0.25, \gamma = 0.2$	0.148429452	0.316858903	0.505288354
	$\lambda = 0.75, \gamma = 0.2$	0.090887414	0.201774827	0.33266224
	$\lambda = 0.25, \gamma = 0.7$	0.049999962	0.119999924	0.209999886
	$\lambda = 0.75, \gamma = 0.7$	0.02541195	0.070823899	0.136235849
Picard		0.0003	0.002328	0.007369875
Runga-Kutta		0.000000435	0.000000081	0.00000046
Euler		0.01	0.020910977	0.032659931
Modified Krasnoselskii Iteration	$\lambda = 0.0001$	0.000311511	0.002331767	0.007378137
	$\lambda = 0.4$	0.003007729	0.012791855	0.030099087
	$\lambda = 0.5$	0.003641293	0.01527775	0.03577398
	$\lambda = 0.9$	0.008516208	0.035355368	0.082333947
New Modified Ishikawa Iteration	$\lambda = 0.5, \gamma = 0.5$	0.075625	0.17125	0.286875
	$\lambda = 0.5, \gamma = 0.25$	0.038515625	0.09703125	0.175546875
	$\lambda = 0.25, \gamma = 0.5$	0.047890624	0.11578125	0.203671875
Extra Modified Ishikawa Iteration	$\lambda = 0.5, \gamma = 0.5$	0.075625	0.17125	0.286875
	$\lambda = 0.5, \gamma = 0.25$	0.038515625	0.09703125	0.175546875
	$\lambda = 0.25, \gamma = 0.5$	0.095546875	0.01890625	0.133359375

After the necessary calculation which is done above, the comparison is shown schematically in Fig.1.

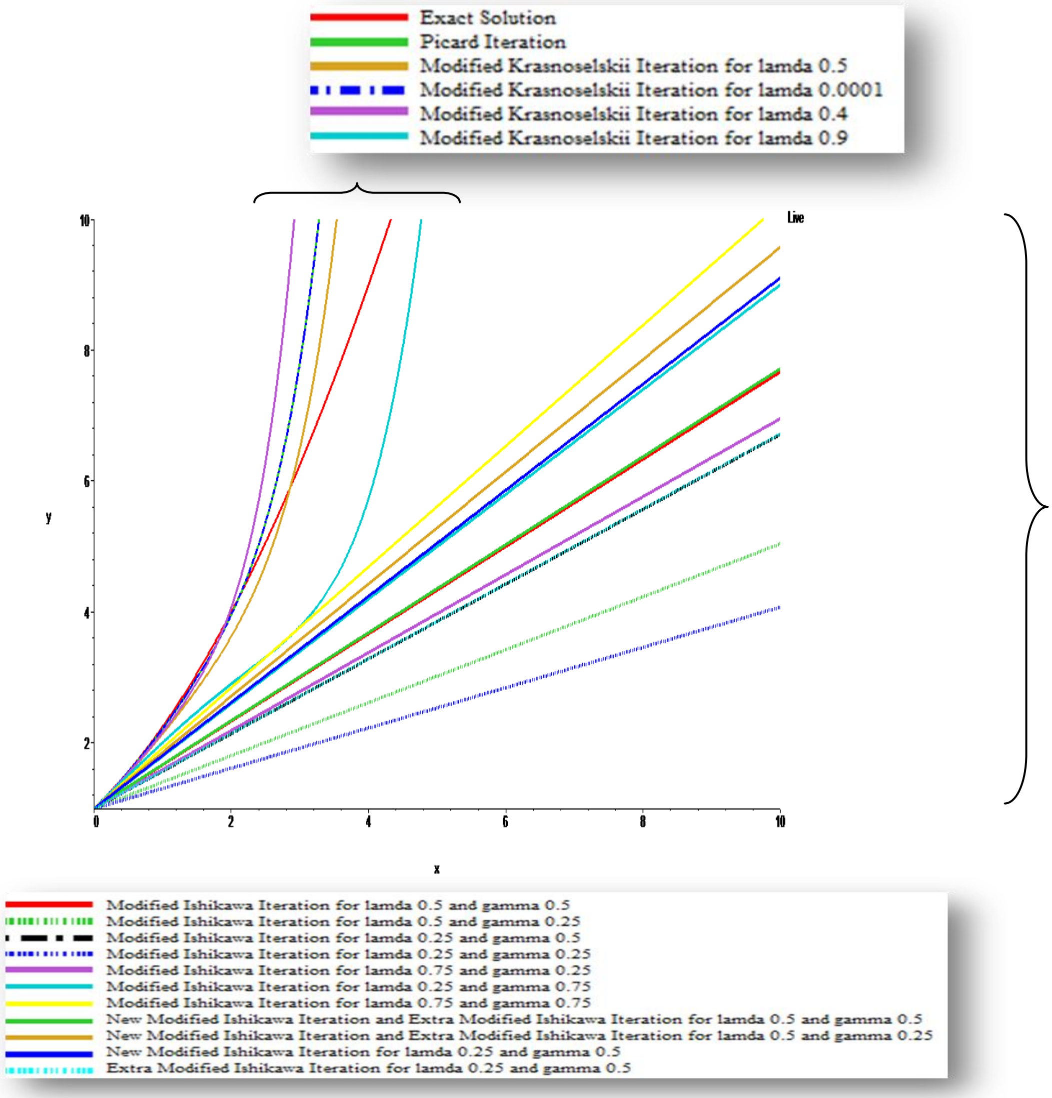

Fig. 1 The comparison of the exact solution and approximate solution of the Example 2.1 for different value of λ and γ.

Corollary 2.1. Absolute error of the Modified Krasnoselskii iteration method is computed taking different values of λ is more effective than Euler, Modified Ishikawa iteration method, New Modified Ishikawa iteration method and Extra Modified Ishikawa iteration method but not better than Runga-Kutta and Picard iteration methods. Absolute error of the Modified Ishikawa iteration method is computed taking different values of λ and γ, which is not more effective than Runga-Kutta, Picard and Euler iteration methods. On the other hand, the New Modified Ishikawa Iteration Method and Extra Modified Ishikawa Iteration Method are compared with the results of absolute errors for Runge-Kutta and Euler methods with the exact solution. And we pointed out that these two methods have more sensitive solution than Modified Ishikawa Iteration method and also indicated that these methods are taken the same value for $(\lambda = 0.5 \text{ and } \gamma = 0.5)$ and $(\lambda = 0.5 \text{ and } \gamma = 0.25)$.

In the conclusion, the comparisons indicate that there is a very good agreement between the numerical solution and the exact solution in terms of accuracy. The result shows that some of the iteration methods are very effective and convenient for solving different type of the equations having the initial conditions with respect to the other methods in the literature.

Example 2.2. Let us consider the differential equation $y' = 2x(y+1)$ subject to the initial condition $y(0) = 0$ [7]. Using Theorem 1.1 and Corollary 1.2, since $T = \int_{x_0}^{x} F(t, y_n(t))dt$ then T has a unique fixed point, which is the unique solution of the differential equation $y' = 2x(y+1)$ with the initial condition $y(0) = 0$.

Firstly, we obtained the exact solution of the equation as $y = e^{x^2} - 1$ .Then we approach the approximate solution using by Picard iteration method as follows:

$y_1 = x^2$

$y_2 = x^2 + \frac{x^4}{2}$

$y_3 = x^2 + \frac{x^4}{2} + \frac{x^6}{6}$

$y_4 = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}$

Now we calculate the approximate solution using by Euler method with $F(x,y) = 2x(y+1)$, $h = 0.2$ and $x_0 = 0$, $y_0 = 0$ which subject to the initial condition $y(0) = 0$ .So we proceed with the calculations as follows:

$F_0 = F(0,0) = 0$

$y_1 = y_0 + hF_0 = 0$

$F_1 = F(0.2,0) = 0.4$

$y_2 = 1.2 + 0.2F_1 = 0.08$

$F_2 = F(0.4,0.08) = 0.864$

$y_3 = 0.08 + 0.2F_2 = 0.2528$

Finally, applying the Runga-Kutta method to the given initial value problem, we carry out the intermediate calculations in each step to give figures after the decimal point and round off the final results at each step to four such places.

Here $F(x,y) = 2x(y+1)$, $x_0 = 0$, $y_0 = 0$ and $h = 0.2$. Using these quantities, we calculated successively k_1, k_2, k_3, k_4 and K_0 which are defined before. Thus, we find k_1, k_2, k_3, k_4 for $n = 0$ as follows:

$k_1 = 0$

$k_2 = 0.04$

$k_3 = 0.0408$

$k_4 = 0.083264$

So $y_1 = 0.040810666$ is obtained for $x_1 = 0.2$. On the other hand, we calculated k_1, k_2, k_3, k_4 for $n = 1$ as follows:

$k_1 = 0.083264853$

$k_2 = 0.129893171$

$k_3 = 0.13269087$

$k_4 = 0.187760245$

Hence $y_2 = 0.173509529$ is calculated for $x_2 = 0.4$. Finally we get k_1, k_2, k_3, k_4 for $n = 2$ as follows:

$k_1 = 0.187761524$

$k_2 = 0.253478058$

$k_3 = 0.260049711$

$k_4 = 0.344054217$

Thus $y_3 = 0.433321409$ is obtained for $x_3 = 0.6$. Now, applying the New Modified Ishikawa Iteration Method to the equation for different value of λ and γ, then

$$y_1 = x^2$$

$y_2 = 0.5x^2$
$y_3 = 0.75x^2$
$y_4 = 0.625x^2$ for $\lambda = 0.5$ and $\gamma = 0.5$
$y_5 = 0.6875x^2$
$y_6 = 0.65625x^2$
$y_7 = 0.671875x^2$

$$y_1 = x^2$$

$y_2 = 0.75x^2$
$y_3 = 0.875x^2$
$y_4 = 0.84375x^2$ for $\lambda = 0.5$ and $\gamma = 0.25$
$y_5 = 0.859375x^2$
$y_6 = 0.85546875x^2$
$y_7 = 0.857421875x^2$

$$y_1 = x^2$$

$y_2 = 0.5x^2$
$y_3 = 0.875x^2$
$y_4 = 0.6875x^2$ for $\lambda = 0.25$ and $\gamma = 0.5$
$y_5 = 0.828125x^2$
$y_6 = 0.7578125x^2$
$y_7 = 0.810546875x^2$

are obtained, and also Extra Modified Ishikawa Iteration Method to the equation for different value of γ and λ, then

$$y_1 = x^2$$

$y_2 = 0.5x^2$
$y_3 = 0.75x^2$
$y_4 = 0.625x^2$ for $\lambda = 0.5$ and $\gamma = 0.5$
$y_5 = 0.6875x^2$
$y_6 = 0.65625x^2$
$y_7 = 0.671875x^2$

$$y_1 = x^2$$

$y_2 = 0.75x^2$
$y_3 = 0.875x^2$
$y_4 = 0.84375x^2$ for $\lambda = 0.5$ and $\gamma = 0.25$
$y_5 = 0.859375x^2$
$y_6 = 0.85546875x^2$
$y_7 = 0.857421875x^2$

$$y_1 = x^2$$

$y_2 = 0.5x^2$
$y_3 = 0.625x^2$
$y_4 = 0.5625x^2$ for $\lambda = 0.25$ and $\gamma = 0.5$
$y_5 = 0.578125x^2$
$y_6 = 0.5703125x^2$
$y_7 = 0.572265625x^2$

are calculated.

Now, let us give the absolute error table of Example 2.2 for different value of λ and γ as follows:

Table 2 Absolute Error

		$x = 0.2$	$x = 0.4$	$x = 0.6$
Modified Ishikawa Iteration	$\lambda = 0.5\,, \gamma = 0.5$	0.014131067	0.066792042	0.193212053
	$\lambda = 0.5\,, \gamma = 0.25$	0.024632796	0.108798958	0.287727608
	$\lambda = 0.25\,, \gamma = 0.5$	0.017953108	0.082080207	0.22761042
	$\lambda = 0.25\,, \gamma = 0.25$	0.028496655	0.124254392	0.322502337
	$\lambda = 0.75\,, \gamma = 0.25$	0.016988257	0.078220802	0.218926758
	$\lambda = 0.25\,, \gamma = 0.75$	0.008810767	0.045510383	0.145329346
	$\lambda = 0.75\,, \gamma = 0.75$	0.003886985	0.025815712	0.101015305
Picard		0.000000002	0.000000899	0.000053574
Runga-Kutta		0.000000108	0.000001342	0.000008005
Euler		0.040810774	0.093510871	0.180529414
Modified Krasnoselskii Iteration	$\lambda = 0.01$	0.000019984	0.000837604	0.00913569
	$\lambda = 0.5$	0.000260774	0.004710871	0.02874414
	$\lambda = 0.9$	0.000665974	0.011194071	0.06156534
New Modified Ishikawa Iteration	$\lambda = 0.5\,, \gamma = 0.5$	0.013935774	0.066010871	0.191454414
	$\lambda = 0.5\,, \gamma = 0.25$	0.006513899	0.036323371	0.124657539
	$\lambda = 0.25\,, \gamma = 0.5$	0.008388899	0.043823371	0.141532539
Extra Modified Ishikawa Iteration	$\lambda = 0.5\,, \gamma = 0.5$	0.013935774	0.066010871	0.191454414
	$\lambda = 0.5\,, \gamma = 0.25$	0.006513899	0.036323371	0.124657539
	$\lambda = 0.25\,, \gamma = 0.5$	0.017920149	0.081948371	0.227313789

After the following absolute Error Table 2 and the necessary calculation which is done above, the comparison is shown schematically in Fig.2.

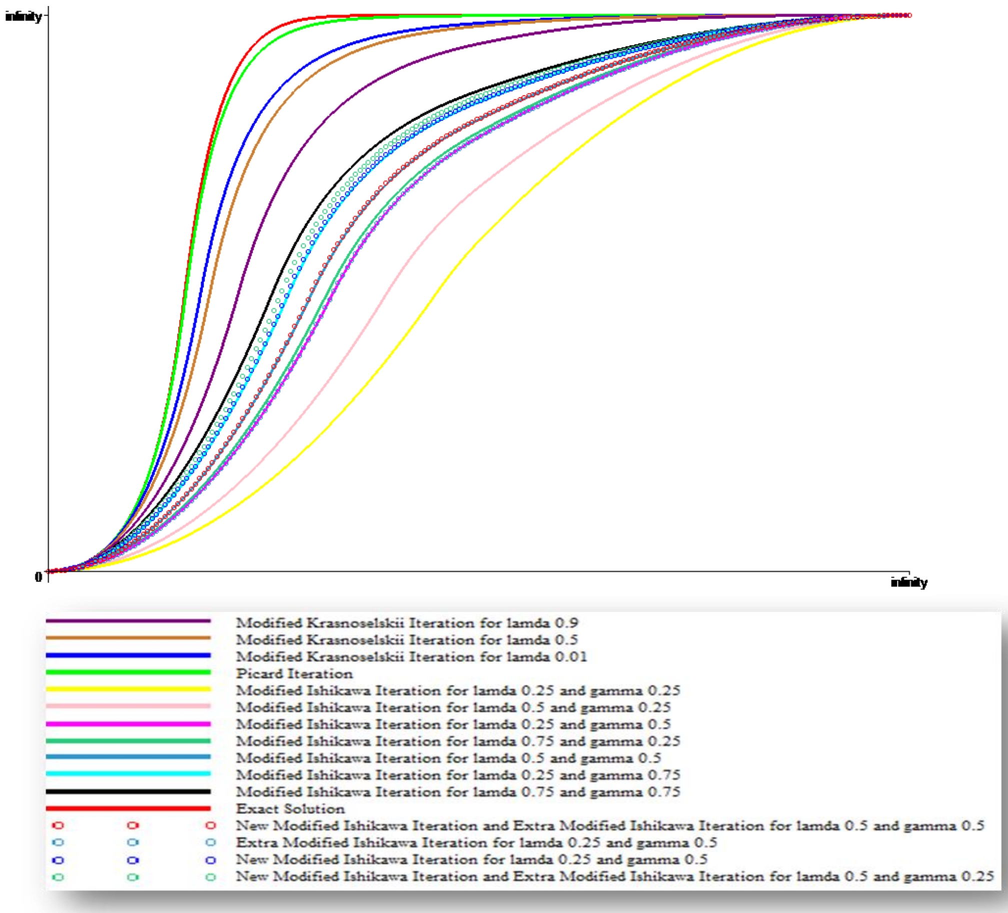

Fig. 2: The comparison of the exact solution and approximate solution of the Example 2.1 for different value of λ and γ.

Corollary 2.2. Absolute error of the Modified Krasnoselskii iteration method is computed taking different values of λ which is more effective than Euler, Modified Ishikawa iteration, New Modified Ishikawa iteration and Extra Modified Ishikawa iteration method but not better than Runga-Kutta and Picard iteration method. On the other hand, the New Modified Ishikawa Iteration Method and Extra Modified Ishikawa Iteration Method compared with the results of absolute errors for Runge-Kutta methods, Euler methods and the exact solution. In the conclusion, the comparisons indicated that there is a very good agreement between the numerical solution and the exact solution in terms of accuracy. The result shows that the New Modified Ishikawa Iteration Method and also Extra Modified Ishikawa Iteration Method are more sensitive than Modified Ishikawa iteration but not more effective than Modified Krasnoselskii Iteration.

Additionally it is observed that New Modified Ishikawa Iteration and Extra Modified Iteration method are taken same value for $(\lambda = 0.5 \text{ and } \gamma = 0.5)$ and $(\lambda = 0.5 \text{ and } \gamma = 0.25)$.

3. CONCLUSION

In this paper, we applied Picard iteration, Modified Krasnoselskii, Modified Ishikawa, Extra Modified Ishikawa and New Modified Ishikawa iteration methods selecting to the different type of the examples and also compared the results of absolute errors for Runga-Kutta and Euler methods with the exact solution. In the conclusion, the comparisons indicated that there is a very good agreement between the numerical solution and the exact solution in terms of accuracy.

The main result shows that some of the methods among the Modified Krasnoselskii, Modified Ishikawa, Extra Modified Ishikawa and New Modified Ishikawa Iteration methods are sometimes very effective and convenient compared with the other methods not only we used but also in the literature as well.

REFERENCES

[1] Agarwal, RP, Meehan, M, O'Regan, D: Fixed Point Theory and Applications, Vol 141, Cambridge University Press, (2004).

[2] Ahmed, AE-S: Convergence of some doubly Sequences Iterations with Errors in Banach spaces, Glob. J. Sci. Front. Res. **105**, 65-69 (2010).

[3] Berinde, V: Iterative Approximation of Fixed Points, 2nd Revised and Enlarged ed. Lecture Notes in Mathematics Berlin: Springer. (1912).

[4] Berinde, V: Iterative Approximation of Fixed Points. North University of Baia Mare, Romania, 18 Juin (2007).

[5] Bildik N, Bakır Y, Mutlu A: The Comparison and Successive Iteration of Approximate Solution of Ordinary Differential Equation with Initial Condition by Modified Krasnoselskii Iteration Method, SCIENTIA-D-12-00620R2 for Scientia Iranica (In press).

[6] Bildik N, Bakır Y, Mutlu A: The New Modified Ishikawa Iteration Method for the Approximate Solution of Different Types of Differential Equations, Fixed Point Theory and Applications 2013, 2013:52, 1687-1812.

[7] Boyce, WE, and Diprima, RC: Elementary Differential Equations and Boundary Value Problems. John Wiley New York (1976).

[8] Brouwer, LEJ: On Continuous one-to-one Transformations of Surfaces into Themselves. Proc. K. Ned. Akad. Wet. 11, 788-798 (1909), Proc. K. Ned. Akad. Wet. 12, 286-297 (1910), Proc. K. Ned. Akad. Wet. 13, 767-774 Proc. K. Ned. Akad. Wet. 14, 300-310 (1912).,Proc. K. Ned. Akad. Wet. 15, 352-360 (1913), Proc. K. Ned. Akad. Wet. 22, 811-814 (1920), Proc. K. Ned. Akad. Wet. 23, 232-234 (1921).

[9] Buong, N, Land, ND: Iteration Methods for Fixed Point of a Nonexpansive Mapping. Int. Math. Forum 6(60), 2963-2974 (2011).

[10] Ciric, LB, Ume, JS: Ishikawa Iterative Process with Errors for Nonlinear Equations of Generalized Monotone type in Banach spaces. Math. Nachr. 27810, 1137-1146 (2005).

[11] Isac, G, Li, J: The Convergence Property of Ishikawa Iteration Schemes in Non-compact Subsets of Hilbert spaces and its Applications to **Complementary** Theory. Comput. Math. Appl 47, 1745-1751 (2004).

[12] Ishikawa, S: Fixed Points and Iterations of a Nonexpansive Mapping in a Banach space. Proc. Am. Math. Soc 59(1), 65-71 (1974).

[13] Ishikawa, S: Fixed points by a New Iteration Method, Proc. Am. Math. Soc. 44(1),147-150 (1974).

[14] Panyanak, B: Mann and Ishikawa Iterative Processes for Multi-valued Mappings in Banach spaces. Comput. Math. Appl. 54(6), 872–877 (2007).

[15] Ross, SL: Differential Equations. Wiley New York (1984).

[16] Shahzad, N, Zegeye, H: On Mann and Ishikawa Iteration Schemes for Multi-valued maps in Banach spaces. Nonlinear Anal., Theory Methods Appl. 71(3), 838-844 (2009).

[17] Shang, X, Qin, M, Su, Y: Strong Convergence of Ishikawa Iterative Method for Nonexpansive Mappings in Hilbert spaces. J. Math. Inequal. 1(2), 195-204 (2007).

[18] Soltuz, SM: The Equivalence of Picard, Mann and Ishikawa iterations dealing with Quasi-Contractive Operators. Math. Commun. 10(1), 81-88 (2005).

[19] Deniz, Sinan, and Necdet Bildik. "Comparison of Adomian decomposition method and Taylor matrix method in solving different kinds of partial differential equations." *International Journal of Modeling and Optimization* 4.4 (2014): 292.

[20] Bildik, Necdet, and Sinan Deniz. "Implementation of Taylor collocation and Adomian decomposition method for systems of ordinary differential equations." AIP Conference Proceedings. Vol. 1648. 2015.

[21] Bildik, Necdet, and Sinan Deniz. "Applications of Taylor collocation method and Lambert W function to systems of delay differential equations." *TJMCS, Article ID* 20130033 (2013): 13.

[22] Bildik, N., and S. Deniz. "Comparison of solutions of systems of delay differential equations using Taylor collocation method, Lambert W function and variational iteration method." *Scientia Iranica. Transaction D, Computer Science & Engineering, Electrical* 22.3 (2015): 1052.

[23] Bildik, Necdet, and Sinan Deniz. "On the Asymptotic Stability of Some Particular Differential Equations." *International Journal of Applied Physics and Mathematics* 5.4 (2015): 252